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A conjectured *R*-matrix

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Abstract. A new spectral parameter independent R-matrix (that depends on all of the dynamical variables) is proposed for the elliptic Calogero–Moser models. The necessary and sufficient conditions for the existence this R-matrix reduces to a determinantal equality involving elliptic functions. The required identity appears new and is still unproven in full generality; we present it as conjecture.

1. Introduction

This paper concerns the construction of an R-matrix for the elliptic Calogero–Moser models. Together with a Lax pair, the R-matrix is a key ingredient of the modern approach to completely integrable systems. In this approach the Lax equation $\dot{L} = [L, M]$ enables us to construct conserved quantities such as the traces $\mathrm{Tr}\, L^k$, while the R-matrix shows that these quantities Poisson commute. A system is said to be completely integrable when we have enough independent, mutually Poisson commuting conserved quantities. For such systems R-matrices must exist [3]. For completely integrable systems Liouville's theorem [1, 12] tells us that we may integrate the equations of motion by quadratures; with certain completeness assumptions‡ on the flows, Arnold's extension of Liouville's theorem ensures the existence of global action-angle variables. The R-matrix is also an essential ingredient when examining the separation of variables of such integrable systems [11, 19].

Recent work has yielded necessary and sufficient conditions for an R-matrix to exist, together with an explicit construction, and we shall now apply this to the elliptic Calogero–Moser models. For the rational and trigonometric degenerations of these models Avan and Talon [2] have constructed R-matrices under an assumption of momentum independence; R-matrices can in principle be functions of the dynamical variables. For the elliptic models, however, [5] shows that no momentum independent R-matrices can be constructed for more than three particles. This restriction can be circumvented by considering R-matrices depending on a spectral parameter, and such momentum independent R-matrices were found for the elliptic Calogero–Moser models by Sklyanin [18] and Braden and Suzuki [5]. A question, however, remains unanswered: are there spectral parameter independent R-matrices for the elliptic Calogero–Moser models? Here we propose such R-matrices. The necessary and sufficient conditions for the R-matrix to exist reduce to a single identity involving matrices with elliptic function entries. This identity appears to be new and we have been unable to prove it in generality: it is given here as conjecture.

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‡ Flaschka [10] gives several simple examples where these assumptions fail.

An outline of the paper is as follows. In sections 2 and 3 we briefly review the construction of R-matrices and the Calogero–Moser models, respectively. In section 4 we combine this material to obtain necessary and sufficient conditions for a spectral parameter independent R-matrix for the elliptic Calogero–Moser models to exist, specifying the R-matrix when such holds true. The necessary and sufficient conditions may be expressed as an equality between two determinants involving elliptic functions: this is presented in section 5. The final section is devoted to a brief discussion.

2. The construction

Recent advances in the construction and understanding of R-matrices follow from the study of a more general matrix equation [6]

$$A^{\mathrm{T}}X - X^{\mathrm{T}}A = B. \tag{1}$$

As we shall review, the R-matrix equation is a particular example of this. Because A is in general singular the general solution to (1) is in terms of a generalized inverse G satisfying

$$AGA = A \qquad \text{and} \qquad GAG = G. \tag{2}$$

Such a generalized inverse† always exists. Given a G satisfying (2) we have at hand projection operators $P_1 = GA$ and $P_2 = AG$ which satisfy

$$AP_1 = P_2A = A \qquad P_1G = GP_2 = G. \tag{3}$$

The matrix equation (1) then has solutions if and only if

(C1) $\quad B^{\mathrm{T}} = -B$

(C2) $\quad (1 - P_1^{\mathrm{T}})B(1 - P_1) = 0$

in which case the general solution is

$$X = \tfrac{1}{2}G^{\mathrm{T}}BP_1 + G^{\mathrm{T}}B(1 - P_1) + (1 - P_2^{\mathrm{T}})Y + (P_2^{\mathrm{T}}ZP_2)A \tag{4}$$

where Y is arbitrary and Z is only constrained by the requirement that $P_2^{\mathrm{T}}ZP_2$ is symmetric. Although the general solution appears to depend on the generalized inverse G, any other choice of generalized inverse will only change the solution within the ambiguities given by (4).

The classical R-matrix construction [17] arises as a particular case of (4) as follows. Suppose the Lax matrix L is in a representation E of a Lie algebra $\mathfrak{g}$ (here taken to be semi-simple). The classical R-matrix is an $E \otimes E$ valued matrix such that

$$[R, L \otimes 1] - [R^{\mathrm{T}}, 1 \otimes L] = \{L \overset{\otimes}{,} L\}. \tag{5}$$

Let T_μ denote a basis for the (finite-dimensional) Lie algebra $\mathfrak{g}$ with $[T_\mu, T_\nu] = c_{\mu\nu}^{\lambda}T_\lambda$ defining the structure constants of $\mathfrak{g}$. Set $\phi(T_\mu) = X_\mu$, where ϕ yields the representation E of the Lie algebra $\mathfrak{g}$; we may take this to be a faithful representation. With $L = \sum_\mu L^\mu X_\mu$, the left-hand side of (5) becomes

$$\{L \overset{\otimes}{,} L\} = \sum_{\mu,\nu}\{L^\mu, L^\nu\}X_\mu \otimes X_\nu$$

† Accounts of generalized inverses may be found in [4, 9, 15, 16]. Indeed the Moore–Penrose inverse—which is unique and always exists—further satisfies $(AG)^\dagger = AG$, $(GA)^\dagger = GA$.

whereas setting $R = R^{\mu\nu} X_\mu \otimes X_\nu$ and $R^{\mathrm{T}} = R^{\nu\mu} X_\mu \otimes X_\nu$ the right-hand side yields

$$\begin{aligned}[R, L \otimes 1] - [R^{\mathrm{T}}, 1 \otimes L] &= R^{\mu\nu}([X_\mu, L] \otimes X_\nu - X_\nu \otimes [X_\mu, L]) \\ &= R^{\mu\nu} L^\lambda([X_\mu, X_\lambda] \otimes X_\nu - X_\nu \otimes [X_\mu, X_\lambda]) \\ &= (R^{\tau\nu} c^\mu_{\tau\lambda} L^\lambda - R^{\tau\mu} c^\nu_{\tau\lambda} L^\lambda) X_\mu \otimes X_\nu.\end{aligned}$$

By identifying $A^{\mu\nu} = c^\nu_{\mu\lambda} L^\lambda \equiv -ad(L)^\nu_\mu$, $B^{\mu\nu} = \{L^\mu, L^\nu\}$ and $X^{\mu\nu} = R^{\mu\nu}$ we see that (5) is an example of (1).

In the R-matrix context, matrix B is manifestly antisymmetric because of the antisymmetry of the Poisson bracket and so (C1) is clearly satisfied. We have thus reduced the existence of an R-matrix to the single consistency equation (C2) and the construction of a generalized inverse to $ad(L)$.

The construction of a generalized inverse for (a generic) $ad(L)$ was given in [7]. Let X_μ denote a Cartan–Weyl basis for the Lie algebra $\mathfrak{g}$. That is, $\{X_\mu\} = \{H_i, E_\alpha\}$, where $\{H_i\}$ is a basis for the Cartan sub-algebra $\mathfrak{h}$ and $\{E_\alpha\}$ is the set of step operators (labelled by the root system Φ of $\mathfrak{g}$). The structure constants are found from

$$[H_i, E_\alpha] = \alpha_i E_\alpha \qquad [E_\alpha, E_{-\alpha}] = \alpha^\vee \cdot H$$

and

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \qquad \text{if } \alpha + \beta \in \Phi.$$

Here $N_{\alpha,\beta} = c^{\alpha+\beta}_{\alpha\beta}$. With these definitions we then have that

$$ad(L) = \begin{array}{c c} & \begin{array}{cc} j & \beta \\ \downarrow & \downarrow \end{array} \\ \begin{array}{c} i \to \\ \alpha \to \end{array} & \begin{pmatrix} 0 & -\beta_i^\vee L^{-\beta} \\ -\alpha_j L^\alpha & \Lambda^\alpha_\beta \end{pmatrix} \end{array} = \begin{pmatrix} 0 & u^{\mathrm{T}} \\ v & \Lambda \end{pmatrix} \tag{6}$$

where we index the rows and columns first by the Cartan sub-algebra basis $\{i, j : 1 \ldots \text{rank } \mathfrak{g}\}$ then the root system $\{\alpha, \beta \in \Phi\}$. We will use this block decomposition of matrices throughout. Here u and v are $|\Phi| \times \text{rank } \mathfrak{g}$ matrices and we have introduced the $|\Phi| \times |\Phi|$ matrix

$$\Lambda^\alpha_\beta = \alpha \cdot L \delta^\alpha_\beta + c^\alpha_{\alpha-\beta\beta} L^{\alpha-\beta} \tag{7}$$

where $\alpha \cdot L = \sum_{i=1}^{\text{rank } \mathfrak{g}} \alpha_i L^i$. With these definitions from [7] we have that for generic L the matrix Λ is invertible and a generalized inverse of $ad(L)$ is given by

$$\begin{pmatrix} 1 & 0 \\ -\Lambda^{-1} v & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & -u^{\mathrm{T}} \Lambda^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda^{-1} \end{pmatrix}. \tag{8}$$

We may now assemble these results to yield the R-matrix. It is convenient to express the Poisson brackets of the entries of L in the same block form in the Cartan–Weyl basis

$$B = \begin{pmatrix} \zeta & -\mu^{\mathrm{T}} \\ \mu & \phi \end{pmatrix} = -B^{\mathrm{T}} \tag{9}$$

where $B^{\alpha j} = \{L^\alpha, L^j\} = \mu_{\alpha j}$ and so on. From the fact that $A = -ad(L)^{\mathrm{T}}$, a generalized inverse of A is given by minus the transpose of the generalized inverse (8) and consequently we obtain the projectors

$$P_1 = \begin{pmatrix} 0 & 0 \\ \Lambda^{-1T} u & 1 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0 & v^{\mathrm{T}} \Lambda^{-1\mathrm{T}} \\ 0 & 1 \end{pmatrix}.$$

The constraint (C2) is now (the rank $\mathfrak{g}\times$ the rank $\mathfrak{g}$ matrix equation)

$$\text{(C2)}\qquad 0=(1-P_1^{\mathrm{T}})B(1-P_1)\equiv\zeta+\mu^{\mathrm{T}}\Lambda^{-1\mathrm{T}}u-u^{\mathrm{T}}\Lambda^{-1}\mu+u^{\mathrm{T}}\Lambda^{-1}\phi\Lambda^{-1\mathrm{T}}u. \tag{10}$$

Supposing the constraint (C2) is satisfied we then find from (4) that the general R-matrix takes the form

$$R=\begin{pmatrix}0 & 0\\ -\Lambda^{-1}\mu+\frac{1}{2}\Lambda^{-1}\phi\Lambda^{-1\mathrm{T}}u & -\frac{1}{2}\Lambda^{-1}\phi\end{pmatrix}+\begin{pmatrix}p & q\\ -\Lambda^{-1}vp-Fu & -\Lambda^{-1}vq-F\Lambda^{\mathrm{T}}\end{pmatrix}. \tag{11}$$

The second term characterizes the ambiguity in R where we have parametrized the matrices Y, Z in (4) by

$$Y=\begin{pmatrix}p & q\\ r & s\end{pmatrix}\qquad\text{and}\qquad Z=\begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

Here the matrices p, q are arbitrary while the entries of Z are such that

$$F=\Lambda^{-1}vav^{\mathrm{T}}\Lambda^{-1\mathrm{T}}+d+\Lambda^{-1}vb+cv^{\mathrm{T}}\Lambda^{-1\mathrm{T}} \tag{12}$$

is symmetric.

3. The Calogero–Moser models

We now recall the salient features of the Calogero–Moser models and in particular those associated with gl_n. For any root system [13, 14] the Calogero–Moser models are the natural Hamiltonian systems

$$H=\frac{1}{2}\sum_i p_i^2+\sum_{\alpha\in\Phi}U(\alpha\cdot x) \tag{13}$$

where (up to a constant) the potential $U(z)$ is the Weierstrass $\wp$-function or a degeneration that will be specified below. For the root systems of the classical algebras a Lax pair may be associated with the models; in the exceptional setting the existence of a Lax pair is still an open question. In fact we do not have direct proof of the complete integrability of the Calogero–Moser models associated with any of the exceptional simple Lie algebras. Let us consider Lax pairs of the following form [8]

$$L=p\cdot H+\sum_{\alpha\in\Phi}f^{\alpha}E_{\alpha}\qquad M=b\cdot H+\sum_{\alpha\in\Phi}w^{\alpha}E_{\alpha} \tag{14}$$

and where the functions f^{α}, w^{α} ($\alpha\in\Phi$) are such that

$$f^{\alpha}=f^{\alpha}(\alpha\cdot x)\qquad w^{\alpha}=w^{\alpha}(\alpha\cdot x). \tag{15}$$

Then

$$\dot{L}=\dot{p}\cdot H+\sum_{\alpha\in\Phi}\alpha\cdot\dot{x}\,f^{\alpha\prime}E_{\alpha} \tag{16}$$

and

$$[L,M]=\sum_{\alpha\in\Phi}((\alpha\cdot pw^{\alpha}-\alpha\cdot bf^{\alpha})E_{\alpha}-f^{-\alpha}w^{\alpha}\alpha^{\vee}\cdot H)+\sum_{\substack{\beta,\gamma\in\Phi\\ \beta+\gamma=\alpha}}c^{\alpha}_{\beta\gamma}f^{\beta}w^{\gamma}E_{\alpha}. \tag{17}$$

We further assume that b is momentum independent. Upon utilizing $\dot{x}_i=p_i$ and comparing (16) and (17) we find that the Lax equation $\dot{L}=[L,M]$ yields the equations of motion for (13) provided the following consistency conditions (for each $\alpha\in\Phi$) are satisfied:

(a)

$$w^\alpha = f^{\alpha\prime}$$

(b)

$$\begin{aligned}\dot{p} &= -\sum_{\alpha\in\Phi} f^{-\alpha} w^\alpha \alpha^\vee = -\sum_{\alpha\in\Phi} f^{-\alpha} f^{\alpha\prime} \alpha^\vee = -\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2}\sum_{\alpha\in\Phi}\frac{2}{\alpha\cdot\alpha} f^{-\alpha} f^\alpha \\ &= -\frac{\mathrm{d}}{\mathrm{d}x}\sum_{\alpha\in\Phi} U(\alpha\cdot x)\end{aligned}$$

(c)

$$\alpha\cdot b = \sum_{\substack{\beta,\gamma\in\Phi\\ \beta+\gamma=\alpha}} c^\alpha_{\beta\gamma}\frac{f^\beta w^\gamma}{f^\alpha} = \sum_{\substack{\beta,\gamma\in\Phi\\ \beta+\gamma=\alpha}} c^\alpha_{\beta\gamma}\frac{f^\beta f^{\gamma\prime}}{f^\alpha}.$$

The second equation determines the potential in terms of the unknown functions f^α. It is the final constraint that is the most difficult to satisfy.

Let us now focus on the Lie algebra gl_n. Here $\Phi = \{e_i - e_j, 1 \leqslant i \neq j \leqslant n\}$, where the e_i form an orthonormal basis of $\mathbb{R}^n$. If e_{rs} denotes the elementary matrix with the (r, s)th entry one and zero elsewhere, then the $n \times n$ matrix representation $H_i = e_{ii}$ and $E_\alpha = e_{ij}$, when $\alpha = e_i - e_j$ gives the usual representation of L. Working with the simple algebra a_n corresponds to the centre-of-mass frame. Here the Calogero–Moser models are built from the functions

$$f^\alpha = \lambda\frac{\sigma(u-\alpha\cdot x)}{\sigma(u)\sigma(\alpha\cdot x)}\,\mathrm{e}^{\zeta(u)\alpha\cdot x}. \tag{18}$$

These functions satisfy the addition formula

$$f^\alpha f^{\beta\prime} - f^\beta f^{\alpha\prime} = (z_\alpha - z_\beta) f^{\alpha+\beta} \tag{19}$$

where

$$z_\alpha = \frac{f^{\alpha\prime\prime}}{2f^\alpha} = \lambda\wp(\alpha\cdot x) + \frac{\lambda}{2}\wp(u). \tag{20}$$

Here $\sigma(x)$ and $\zeta(x) = \sigma'(x)/\sigma(x)$ are the Weierstrass sigma and zeta functions [21]. The quantity u in (18) is known as the spectral parameter. We find

$$U(\alpha\cdot x) = -\frac{\lambda^2}{2}(\wp(\alpha\cdot x) - \wp(u)) \tag{21}$$

and that

$$b = \frac{1}{2(n+1)}\sum_{\substack{\beta,\gamma\in\Phi\\ \beta+\gamma=\alpha}} c^\alpha_{\beta\gamma} z_\beta \alpha$$

which in components becomes

$$b_i = \lambda\sum_{j\neq i}\wp(x_i - x_j). \tag{22}$$

The Weierstrass $\wp$-function includes as degenerations the potentials

$$U(z) = \frac{\lambda^2}{z^2},\quad \frac{\lambda^2}{\sin z^2},\quad \frac{\lambda^2}{\sinh z^2}. \tag{23}$$

The first of these is the original (rational) Calogero–Moser model whereas the second is the Sutherland model [20].

4. The gl_n Calogero–Moser *R*-matrix

We shall now apply the construction of section 2 to the gl_n Calogero–Moser models. Upon examination of (10) and (11) we see that the relevant quantities to calculate are the matrices $B^{\mu\nu} = \{L^\mu, L^\nu\}$ and the components Λ and u of $ad(L)$. Using $\{p_j, f^\alpha\} = \{p_j, \alpha\cdot q\} f^{\alpha\prime} = \alpha_j f^{\alpha\prime}$ we find that

$$\{L \overset{\otimes}{,} L\} = \sum_{\mu,\nu} \{L^\mu, L^\nu\} X_\mu \otimes X_\nu = \sum_{j,\alpha} \alpha_j f^{\alpha\prime}(H_j \otimes E_\alpha - E_\alpha \otimes H_j).$$

This means that we have

$$B = \begin{pmatrix} 0 & \beta_i f^{\beta\prime} \\ -\alpha_j f^{\alpha\prime} & 0 \end{pmatrix} = -B^{\mathrm{T}}$$

and upon comparison with (9) we see that $\zeta = \phi = 0$. For the case in hand

$$u_{\alpha k} = -f^{-\alpha}\alpha_k \qquad \Lambda^\alpha_\beta = \alpha\cdot p\,\delta^\alpha_\beta + c^\alpha_{\alpha-\beta\beta} f^{\alpha-\beta} \tag{24}$$

and it will be convenient to introduce the ($|\Phi| \times \operatorname{rank} \mathfrak{g}$) matrix

$$w_{\alpha k} = -f^{\alpha\prime}\alpha_k. \tag{25}$$

Thus $B = \begin{pmatrix} 0 & -w^{\mathrm{T}} \\ w & 0 \end{pmatrix}$. Being quite explicit, if $\alpha = e_i - e_j$ and $\beta = e_r - e_s$ then

$$\begin{aligned} &\Lambda: \quad \Lambda^{(ij)}_{(rs)} = (p_i - p_j)\delta^i_r\delta^j_s + f(x_i - x_r)\delta^j_s - f(x_s - x_j)\delta^i_r \\ &u: \quad u_{(ij),k} = -(\delta_{ik} - \delta_{jk}) f(x_j - x_i) \\ &w: \quad w_{(ij),k} = -(\delta_{ik} - \delta_{jk}) f'(x_i - x_j) \end{aligned} \tag{26}$$

where we adopt the obvious notational shorthand of replacing the matrix indices for $\alpha = e_i - e_j$ by (ij) and so on.

With these quantities at hand the necessary and sufficient condition (C2) given by (10) takes the form

$$\text{(C2)} \qquad 0 = u^{\mathrm{T}}\Lambda^{-1}w - w^{\mathrm{T}}\Lambda^{-1T}u \tag{27}$$

which in components becomes

$$\text{(C2)} \qquad 0 = \sum_{\alpha,\beta} (\alpha_i f^{-\alpha}(\Lambda^{-1})^\alpha_\beta f^{\beta\prime}\beta_j - \alpha_i f^{\alpha\prime}(\Lambda^{-1})^\beta_\alpha f^{-\beta}\beta_j). \tag{28}$$

When this is satisfied we have from (11) that the general *R*-matrix is given by

$$R = \begin{pmatrix} 0 & 0 \\ (\Lambda^{-1})^\alpha_\beta f^{\beta\prime}\beta_j & 0 \end{pmatrix} + \begin{pmatrix} p & q \\ -\Lambda^{-1}vp - Fu & -\Lambda^{-1}vq - F\Lambda^{\mathrm{T}} \end{pmatrix}. \tag{29}$$

The second term, which characterizes the possible ambiguity in the *R*-matrix, was described in section 2.

It is instructive to consider how the minimal solution given by the first term of (29) satisfies (5). We have

$$R^{\alpha j} = (\Lambda^{-1})^\alpha_\beta f^{\beta\prime}\beta_j \qquad R^{ij} = 0 \qquad R^{i\alpha} = 0 \qquad R^{\alpha\beta} = 0. \tag{30}$$

This is to be compared with the previously known *R*-matrix [2]

$$R^{\alpha j} = 0 \qquad R^{ij} = 0 \qquad R^{i\alpha} = -\frac{|\alpha_i|}{2} f^\alpha \qquad R^{\alpha\beta} = \delta_{\alpha+\beta,0}\frac{f^{\alpha\prime}}{f^\alpha}$$

which exists only for the potentials $U(z) = \lambda^2/z^2$, $\lambda^2/\sin z^2$, $\lambda^2/\sinh z^2$ [5]. Examination of the general R-matrix equation (5) yields three different equations depending on the range of indices $\{\mu, \nu\}$. For $(\mu, \nu) = (i, j)$, (i, α) and (α, β), respectively, these are

$$0 = \sum_{\alpha} (R^{\alpha j}\alpha_i - R^{\alpha i}\alpha_j) f^{-\alpha}$$

$$\alpha_i f^{\alpha\prime} = \alpha \cdot p R^{\alpha i} - \sum_j \alpha_j R^{ji} f^{\alpha} + \sum_{\beta} (\beta_i f^{\beta} R^{-\beta\alpha} + f^{\alpha-\beta} R^{\beta i} c^{\alpha}_{\alpha-\beta\beta}) \tag{31}$$

and

$$0 = \sum_i (\alpha_i R^{i\beta} f^{\alpha} - \beta_i R^{i\alpha} f^{\beta}) - (\alpha \cdot p R^{\alpha\beta} - \beta \cdot p R^{\beta\alpha})$$
$$+ \sum_{\gamma} (R^{\gamma\beta} c^{\alpha}_{\gamma\alpha-\gamma} f^{\alpha-\gamma} - R^{\gamma\alpha} c^{\beta}_{\gamma\beta-\gamma} f^{\beta-\gamma}).$$

Employing (30) we see that the final two equations are automatically satisfied. The first equation (31) is less obvious until we realize that it just expresses the remaining constraint (C2) necessary for a solution to exist. This identification follows on from using $R^{\alpha j} = (\Lambda^{-1})^{\alpha}_{\beta} f^{\beta\prime}\beta_j$.

At this stage we have reduced the existence of an R-matrix for Calogero–Moser systems to that of a constraint equation.

Result 1. The elliptic Calogero–Moser system has R-matrix (30) if and only if (28)—or equivalently (27)—is satisfied.

We remark that (27) is again of the form (1) for the (non-square) matrix $\tilde{A} = \Lambda^{-1\mathrm{T}} u$ and $\tilde{B} = 0$, where we now wish to show that $\tilde{X} = w$ is a solution. The general theory applies and as $\tilde{B} = 0$ the constraints are automatically satisfied. One discovers in this situation that the requirement for $P_2^{\mathrm{T}} Z P_2$ to be symmetric is equivalent to the symmetry of the matrix $w^{\mathrm{T}} \Lambda^{-1\mathrm{T}} u$.

5. The constraint

It remains for us to analyse the constraint equation (28). Although the inverse matrices here look somewhat daunting we may use the co-factor expansion of an inverse to give

$$\begin{vmatrix} 0 & k \\ l & \Lambda \end{vmatrix} = -|\Lambda| k^{\mathrm{T}} \Lambda^{-1} l.$$

Thus (28) is equivalent to showing that (for each i, j) the $(|\Phi|+1) \times (|\Phi|+1)$ determinants satisfy

$$\begin{vmatrix} 0 & \alpha_i f^{-\alpha} \\ \beta_j f^{\beta\prime} & \Lambda \end{vmatrix} = \begin{vmatrix} 0 & \alpha_j f^{-\alpha} \\ \beta_i f^{\beta\prime} & \Lambda \end{vmatrix} \tag{32}$$

where† $\Lambda = (\Lambda_{\beta\alpha})$. To be quite explicit we wish to show that (for each i, j)

$$\begin{vmatrix} 0 & u_{(rs),i} \\ w_{(kl),j} & \Lambda^{(kl)}_{(rs)} \end{vmatrix} = \begin{vmatrix} 0 & u_{(rs),j} \\ w_{(kl),i} & \Lambda^{(kl)}_{(rs)} \end{vmatrix} \tag{33}$$

where Λ (to be invertible), u and w are given by (26), the indices (kl), (rs) run over ordered distinct pairs and the functions being considered are given by $f(x) =$

† Note the adjugate matrix of Λ involves the transpose of the co-factors and hence the perhaps puzzling interchange of rows and columns here.

$(\sigma(u-x))/(\sigma(u)\sigma(x))\,\mathrm{e}^{\zeta(u)x}$. Actually, because of the symmetry of the problem, it suffices to show that (32) holds for any two indices $i \neq j$ (it clearly holds for $i = j$) and we may take these, for example, to be $i = 1,\ j = 2$.

We are unable to prove (32) in generality. Symbolic manipulation has verified it true for small numbers of particles and it has satisfied extensive numerical checks. At present we can only present it as conjecture. The conjectured identity appears new.

We remark that in the present setting one can show that for arbitrary functions f^α (for which Λ is invertible)

$$0 = (u^{\mathrm{T}}\Lambda^{-1}v)_{ij} = \sum_{\alpha,\beta} \alpha_i^\vee L^{-\alpha}(\Lambda^{-1})^\alpha_\beta L^\beta \beta_j = \begin{vmatrix} 0 & \alpha_i f^{-\alpha} \\ \beta_j f^\beta & \Lambda \end{vmatrix}. \tag{34}$$

Whereas (34) is true for any functions f^α, equation (32) will only hold for a more restricted class of functions. The constraint requires that functions of the form (18) satisfy (32).

6. Discussion

This paper has been devoted to the construction of a spectral parameter independent R-matrix for the elliptic Calogero–Moser models. Previous work has shown that no momentum independent and spectral parameter independent R-matrix exists for the models for more than three particles. By viewing the R-matrix equation as a particular case of the general matrix equation (1) we are able to give necessary and sufficient conditions for a (generally momentum dependent) R-matrix to exist. No recourse to special ansatz is needed and the general form of the R-matrix can be specified. The elliptic Calogero–Moser model has R-matrix (30) if and only if (28)—or equivalently (27)—is satisfied. The desired R-matrices existence has thus been reduced to the validity of a single constraint. This constraint may equally be cast as the equality between two determinants (32) involving elliptic functions. (We have unpacked most of the Lie algebra notation in the explicit form (33).) Such an identity appears new. Unfortunately we have been unable to prove (32) in generality and we present it here as conjecture.

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